M. Math/ B3 Ist Semester 2009-10 Back paper Exam Topology

Answer all the questions. Each question is worth 10 points.
All topological spaces $(X, \mathcal{T})$ are assumed to be Hausdorff.

1) Give an example of a separable topological space that is not second countable.

2) Let (X, d) be a metric space. Let $f : X \to R$ be a function. Show that the ϵ-δ definition of continuity is equivalent to for any open set $U \subset R$, $f^{-1}(U)$ is an open set.

3) State and prove the Baire category theorem for complete metric spaces.

4) Let (X, d) be a metric space. Show that for any closed set $C \subset X$, there is a continuous function $f : X \to R$ such that $C = f^{-1}(0)$.

5) Let $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha \in \Delta}$ be a family of topological spaces such that the product space $X = \prod X_\alpha$ is connected. Show that each X_α is a connected space.

6) Let X be a compact space and Y a Hausdorff space. Suppose $f : X \to Y$ is a continuous bijection. Show that f^{-1} is also continuous.

7) Give an example with full details of a locally compact metric space that is not compact.

8) Let $A \subset X$ be a dense set. Suppose $f, g : X \to X$ be two continuous functions such that $f(x) = g(x)$ for all $x \in A$. Show that $f = g$.

9) Prove that the fundamental group of RP^2 is $Z/2Z$.

10) If f is a homeomorphism between (X, x) and (Y, y), then show that

$$\pi_1(X, x) \simeq \pi_1(Y, y)$$

Is the converse true - if two spaces have isomorphic fundamental groups then are they homeomorphic ?